Section 6.4: Permutations

In this section we study a useful formula for the number of permutations of n objects taken k at a time. It is really just a special application of the multiplication principle, but the phenomenon occurs often enough in reality that it is useful to pull out the general principles and create a special name and formula for this this situation.

Example Alan, Cassie, Maggie, Seth and Roger are friends who want to take a photograph with three of the five friends in it.

Alan (who likes to be thorough) makes a complete list of all possible ways of lining up 3 out of the 5 friends for a photo as follows:

AMC	AMS	AMR	ACS	ACR
ACM	ASM	ARM	ASC	ARC
CAM	MAS	MAR	CAS	CAR
CMA	MSA	MRA	CSA	CRA
MAC	SAM	RAM	SAC	RCA
MCA	SMA	RMA	SCA	RAC
ASR	MSR	MCR	MCS	CRS
ARS	MRS	MRC	MSC	CSR
SAR	SMR	RMC	CMS	RCS
SRA	SRM	RCM	CSM	RSC
RSA	MRS	CRM	SMC	SCR
RAS	MSR	CMR	SCM	SRC

Alan has just attended a finite math lecture on the multiplication principle and suddenly realizes that their may be an easier way to count the possible photographs.

He reckons he has 5 choices for the position on the left, and

once he's chosen who should stand on the left, he will have 4 choices for the position in the middle

and once he fills both of above positions, he has 3 choices for the one on the right.

This gives a total of $5 \times 4 \times 3 = 60$ possibilities.

Alan has listed all **Permutations** of the five friends taken 3 at a time.

The number of permutations of 5 objects taken 3 at a time has a special symbol:

$$P(5,3)$$

and as we have seen $P(5,3) = 60$.

Definition A **Permutation** of **n** objects taken **k** at a time is an arrangement (Line up, Photo) of k of the n objects in a specific <u>order</u>.

When using the multiplication principle to count the number of such permutations, as Alan did, the following characteristics are key:

1. A permutation is an arrangement or sequence of selections of objects from a single set.
2. Repetitions are not allowed or the same element may not appear more than once in an arrangement. (In the example above, the photo AAA is not possible).

3. the order in which the elements are selected or arranged is significant. (In the above example, the photographs AMC and CAM are different).

The number of such permutations is denoted by the symbol, $P(n, k)$.

Example Calculate $P(10, 3)$, the number of photographs of 10 friends taken 3 at a time.

$P(10, 3) = 10 \cdot 9 \cdot 8$ Note that you start with 10 and multiply 3 numbers.

Example Calculate $P(6, 4)$, the number of photographs of 6 friends taken 4 at a time.

$P(6, 4) = 6 \cdot 5 \cdot 4 \cdot 3$.

We can use the same principles that Alan did to find a general formula for the number of permutations of n objects taken k at a time, which follows from an application of the multiplication principle:

$$\boxed{P(n, k) = n \times (n-1) \times (n-2) \times \cdots \times (n-k+1).}$$

Note that there are k consecutive numbers on the right hand side.

Example In how many ways can you choose a President, secretary and treasurer for a club from 12 candidates, if each candidate is eligible for each position, but no candidate can hold 2 positions? Why are conditions 1, 2 and 3 relevant here?

$P(12, 3)$. Condition 1 is satisfied because we have a single set of 12 candidates for all 3 positions. Condition 2 is satisfied because no one can hold more than one position. Condition 3 is satisfied because being president is different than being treasurer or secretary.

Example You have been asked to judge an art contest with 15 entries. In how many ways can you assign 1st , 2nd and 3rd place? (Express your answer as $P(n, k)$ for some n and k and evaluate.)

$P(15, 3) = 15 \cdot 14 \cdot 13 = 2,730$.

Example Ten students are to be chosen from a class of 30 and lined up for a photograph. How many such photographs can be taken? (Express your answer as $P(n, k)$ for some n and k and evaluate.)

$P(30, 10) = 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21$. Note $30 - 10 = 20$ and we stopped at 21.

$P(30, 10) = 109,027,350,432,000$

Example In how many ways can you arrange 5 math books on a shelf.

$P(5,5) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. Note $5 - 5 = 0$ and we stopped at 1.

$P(30, 10) = 120$

Factorials The numbers $P(n, n) = n \times (n-1) \times (n-2) \times \cdots \times 1$ are denoted by $n!$ or factorial n. We can rewrite our formula for $P(n, k)$ in terms of factorials:

$$\boxed{P(n, k) = \frac{n!}{(n-k)!}.}$$

Example (a) Evaluate 12!

(b) Evaluate $P(12, 5)$.

$12! = P(12, 12) = 12 \cdot 11 \cdots 2 \cdot 1 = 479,001,600$.

$P(12, 5) = \frac{P(12,12)}{P(7,7)} = \frac{479{,}001{,}600}{5{,}040} = 95,040$.

Example In how many ways can 10 people be lined up for a photograph?

$10! = P(10, 10)$.

Example How many three letter words(including nonsense words) can you make from the letters of the English alphabet, if letters cannot be repeated? (Express your answer as $P(n, k)$ for some n and k and evaluate.)

$P(26, 3)$.

Permutations of objects with some alike

In this section, we will only consider permutations of sets of n objects taken n at at time, in other words rearrangements of n objects. We will consider situations in which some objects are the same. Note that if two objects in the arrangement are the same, we get the same arrangement when we switch the two.

Example How many words can we make by rearranging the letters of the word
BEER?

Note that if we switch the two E's in any arrangement, we do not get a new word, so if we count all permutations of 4 letters, we over count the number of words. Thus among the $4! = 24$ arrangements of the 4 letters above, the word EEBR appears twice. Similarly every other word appears twice on the list of 4! arrangements. Thus the number of different words we can form by rearranging the letters must be

$$4!/2 = \frac{4!}{2!}.$$

Note that 2! counts the number of ways we can permute the E's in any given arrangement.

(This may be clearer if we distinguish the two E's in the given word, writing it as BE_1E_2R. The 4! permutations of these letters look like

$$\{E_1E_2BR, \quad E_2E_1BR, \quad BE_1E_2R, \quad BE_2E_1R, \dots\}$$

with every word repeated 2 times.)

In general the number of permutations of n objects with r of the objects identical is $\frac{n!}{r!}$.

Example How many distinct words(including nonsense words) can be made from rearrangements of the word

ALPACA

$\frac{6!}{3!}$. There are 6 letters in ALPACA and one of them, 'A' is repeated 3 times. $\frac{6!}{3!} = \frac{720}{6} = 120$

If a set of n objects contains k subsets of objects in which the objects in each subset are identical and objects in different subsets are not identical, the number of different permutations of all n objects is

$$\frac{n!}{r_1! \cdot r_2! \cdot \dots \cdot r_k!},$$

where r_1 is the number of objects in the first subset, r_2 is the number of objects in the second subset etc.... Note that for a subset of size 1, we have $1! = 1$.

Example How many distinct words(including nonsense words) can be made from rearrangements of the word

BANANA

$\frac{6!}{2! \cdot 3!}$. There are 6 letters in BANANA and one of them, 'A' is repeated 3 times and another B is repeated 2 times. $\frac{6!}{2! \cdot 3!} = \frac{720}{2 \cdot 6} = 60$

Example How many distinct words(including nonsense words) can be made from rearrangements of the word

BOOKKEEPER

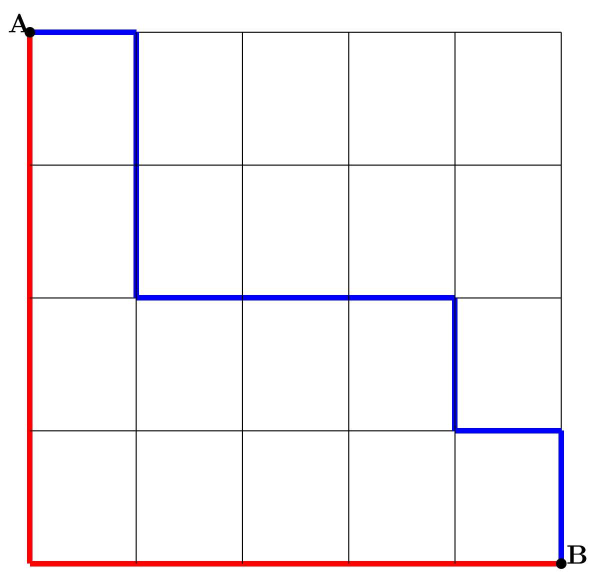
$\dfrac{10!}{1! \cdot 3! \cdot 2! \cdot 2! \cdot 1! \cdot 1!}$. There are 10 letters in BOOKKEEPER. In alphabetical order, B $\leftrightarrow$ 1, E $\leftrightarrow$ 3, K $\leftrightarrow$ 2, O $\leftrightarrow$ 2, P $\leftrightarrow$ 1, R $\leftrightarrow$ 1.

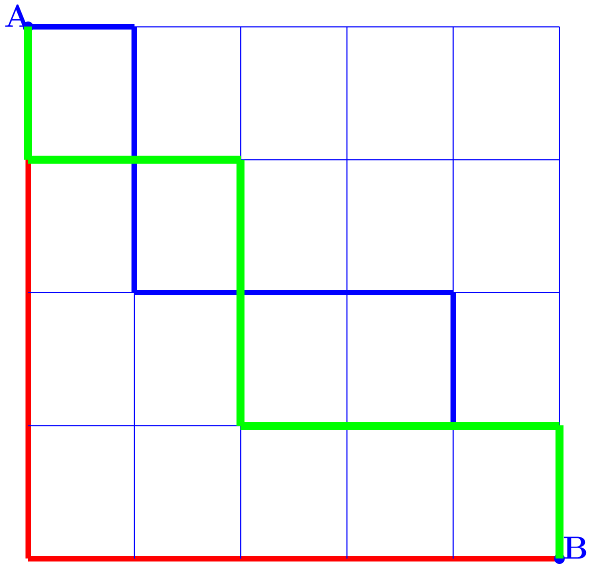
Note that the total number of letters is the sum of the multiplicities of the distinct letters. $\frac{10!}{1! \cdot 3! \cdot 2! \cdot 2! \cdot 1! \cdot 1!} = \frac{3{,}628{,}800}{6 \cdot 2 \cdot 2} = 151,200$.

Taxi cab Geometry

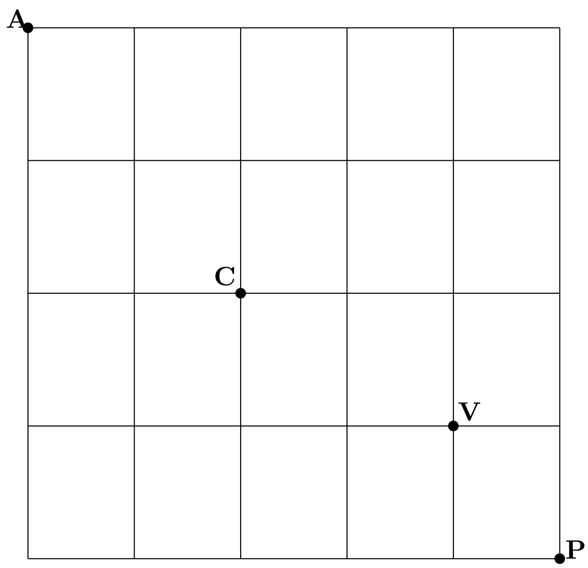
On the street map shown below, any route that a taxi cab can take from the point A to the point B if they always travel south or east can be described uniquely as a sequence of S's and E's (S for South and E for East). To get from A to B the taxi driver must travel south for four blocks and east for five blocks. Any sequence of 4 S's and 5 E's describes such a route and two routes are the same only if the sequences describing them are the same. Thus the number of taxi cab routes from A to B is the number of different rearrangements of the sequence SSSSEEEEE which is $\dfrac{9!}{4!5!}$.

Below we show the sequence SSSSEEEEE in red and the sequence ESSEEESES in blue. Draw the sequence SEESSEEES.





Example A streetmap of Mathville is given below. You arrive at the Airport at A and wish to take a taxi to Pascal's house at P. The taxi driver, being an honest sort, will take a route from A to P with no backtracking, always traveling south or east.



(a) How many such routes are possible from A to P?

You need to go 4 blocks south and 5 blocks east for a total of 9 blocks so the number of routes is $\dfrac{9!}{4! \cdot 5!} = \dfrac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 9 \cdot 2 \cdot 7 =$ 126.

(b) If you insist on stopping off at the Combinatorium at C, how many routes can the taxi driver take from A to P?

This is really two taxicab problems combined with the Multiplication Principle. The answer, in words, is 'the number of paths from A to C' times 'the number of paths from C to P'. The first is $\frac{4!}{2! \cdot 2!}$ and the second is $\frac{5!}{2! \cdot 3!}$ so the answer is $6 \cdot 10 = 60$.

(c) If wish to stop off at both the combinatorium at C and the Vennitarium at V, how many routes can your taxi driver take?

This is three taxicab problem. The answer, in words, is 'the number of paths from A to C' times 'the number of paths from C to V' times 'the number of paths from V to P. The first is $\frac{4!}{2! \cdot 2!}$, the second is $\frac{3!}{1! \cdot 2!}$ and the third is $\frac{2!}{1! \cdot 1!}$ so the answer is $6 \cdot 3 \cdot 2 = 36$.

(d) If you wish to stop off at either C or V(at least one), how many routes can the taxi driver take.

This certainly the most complicated of this set of problems. It involves not only taxis but also the Inclusion-Exclusion Principle. To see this, suppose C denotes the set of all paths from A to P that go through C and that V denotes the set of all paths from A to P that go through V.

The number we want is $n(C \cup V)$ since $C \cup V$ is the set of all paths which go through C or V.

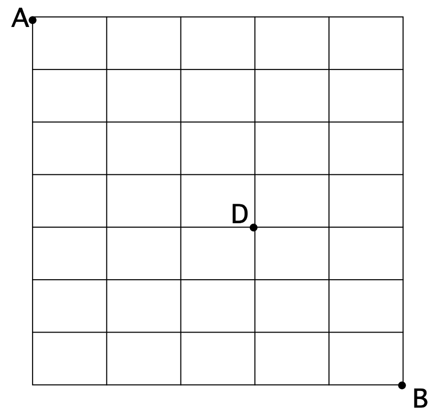
We have already computed $n(C) = 60$. For $n(V)$ we have $n(V) = \frac{7!}{3! \cdot 4!} \cdot \frac{2!}{1! \cdot 1!} = \frac{7 \cdot 6 \cdot 5}{6} \cdot 2 = 70$.

We still need $n(C \cap V)$ but $C \cap V$ is the set of all paths which go through both C and V and we have already computed $n(C \cap V) = 36$.

Hence

$$n(C \cup V) = 60 + 70 - 36 = 94$$

Example (a) Christine, on her morning run, wants to get from point A to point B. How many routes with no backtracking can she take?



(b) How many of those routes go through the point D?

(c) If Christine wants to avoid the Doberman at D, how many routes can she take?